Explicit Attitude Stability Criteria for Passive, Earth-Pointing Satellites

John R. Murphy*
Hughes Aircraft Company, El Segundo, Calif.
and
Thomas R. Kane†
Stanford University, Stanford, Calif.

A method is developed for constructing stability criteria applicable to Earth-pointing motions of elastic, dissipative satellites posessing a finite number of degrees of freedom. The method is explicit in the sense that it leads to inequalities formulated entirely in terms of readily available quantities. In addition, explicit conditions that guarantee pervasive damping are stated. This makes it possible to establish instability or asymptotic stability with relative ease. To illustrate the procedures under consideration, the method is applied to a specific satellite configuration.

I. Introduction

A METHOD for generating stability criteria applicable to spinning, torque-free, deformable satellites was described[1] recently. The present paper deals with a similar approach to the construction of stability criteria for Earth-pointing, deformable satellites in circular orbits. As before, external and internal stability conditions are uncoupled from each other, which can lead to substantial savings in the labor associated with a given problem. In addition, requirements for pervasive damping are set forth explicitly. Thus, using the results here reported, one can predict instability or asymptotic stability after carrying out a number of relatively simple instructions.

The principles upon which this work is based have been employed by many authors. Analyses that bear a particularly close resemblance to the one at hand are those of Pringle,[2,4] Willems,[5,6] and Buckins et al.[7] All of these involve the use of the Hamiltonian as a Liapunov testing function. This approach has been used also to deal with systems containing continuous elements,[8-10] which, at first glance, may appear to fall outside the scope of the present theory, but which can, in fact, be brought under its purview by modeling continuous elements as collections of discrete elements. Finally, numerous specific satellite configurations have been investigated in terms of this methodology. For example, Bainum and Mackison[11] presented stability conditions for a satellite originally proposed by Tinling and Merrick,[12] Nelson and Meirovitch[13] considered a satellite consisting of rigid body and two spring-supported particles; and Bainum et al.[14] obtained stability information for a tethered satellite by these means. However, there also exist other roads leading to explicit conditions for instability or asymptotic stability. For example, Robe[15] used the method of Routhian arrays[16] in connection with a tethered satellite; the Routh-Hurwitz[17] technique was applied to the Vertistat satellite[18]; and a satellite carrying a two-degree-of-freedom boom was analyzed recently[19] in terms of the criteria of Lienard and Chipart.[20] The results of the present paper apply to all of these problems.

II. System Description

The system to be studied is a deformable body S composed of particles and rigid bodies. These are connected to each other in such a way that the associated constraint equations are time-independent and holonomic, and such that S possesses n "internal" degrees of freedom; that is, n generalized coordinates $q_1,\ldots,q_n$ suffice for a complete description of the relative disposition of all parts of A. It is assumed that S moves in a Newtonian reference frame N under the action of gravitational forces exerted by a particle E (representing the Earth) of mass μ; and the distance between E and every point of S greatly exceeds the largest dimension of S. Under these circumstances, E may be regarded as fixed in N; the mass center S^* of S can move in an essentially circular orbit of radius ρ, centered at E, and fixed in N; and the angular speed that characterizes this motion is given by $\Omega = (G\mu\rho^{-3})^{1/2}$, where G is the universal gravitational constant.

One can complete the description of such a motion by specifying, in addition to $q_1,\ldots,q_n$, the relative orientation of two dextral sets of orthogonal unit vectors, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$, those of the first set being, respectively, parallel to the line segment $E-S^*$, to the tangent to the orbit, and to the orbit normal, whereas those of the second set are, respectively, parallel to (instantaneous) central principal axes of inertia of S. To this end, we introduce reference frame A, in which $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ are fixed, and reference frame B, in which $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ are fixed, and let θ_1, θ_2, and θ_3 be the radian measures of angles such that one can bring B into a desired orientation in A by aligning $\boldsymbol{b}_k$ with $\boldsymbol{a}_k$ $(k=1,2,3)$ and then performing successive rotations characterized by the vectors $\theta_1\boldsymbol{b}_1$, $\theta_2\boldsymbol{b}_2$, and $\theta_3\boldsymbol{b}_3$. We refer to θ_1, θ_2, and θ_3 as "external" coordinates, whereas $q_1,\ldots,q_n$ are called "internal coordinates.

The only external forces acting on S are the gravitational forces exerted by E. Their contributions to the generalized active forces F_{θ_k} and F_{q_r} are presumed to be given with sufficient accuracy by

$$(F_{\theta_k})_E = -W(\theta,q)_{,k} \qquad (k=1,2,3) \tag{1}$$

$$(F_{q_r})_E = -W(\theta,q)_{,r} \qquad (r=1,\ldots,n) \tag{2}$$

where $W(\theta,q)$ is a function of θ_k and q_r $(k=1,2,3;\ r=1,\ldots,n)$ which can be expressed as[21]

$$W(\theta,q) = -2^{-1}\Omega^2[J_1(q)+J_2(q)+J_3(q)-3\mathbf{a}_1\cdot \boldsymbol{J}(q)\cdot \boldsymbol{a}_1] \tag{3}$$

where $J_1(q)$, $J_2(q)$, and $J_3(q)$ are, respectively, the central principal moments of inertia of S associated with $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$, and $\boldsymbol{J}(q)$ is the central inertia dyadic of S; that is

$$\boldsymbol{J}(q) = J_1(q)\boldsymbol{b}_1\boldsymbol{b}_1 + J_2(q)\boldsymbol{b}_2\boldsymbol{b}_2 + J_3(q)\boldsymbol{b}_3\boldsymbol{b}_3 \tag{4}$$

Received Aug. 19, 1974; revision received Oct. 17, 1975.
Index category: Spacecraft Attitude Dynamics and Control.
*Staff Engineer.
†Professor of Applied Mechanics.

In Eqs. (1) and (2), as throughout the remainder of the paper, a subscript comma followed by one or more letters denotes partial differentiation with respect to external variables when the letter is k or ℓ, and with respect to internal variables when the letter is r or s. If the quantity being differentiated is a vector function, a left superscript designates the reference frame in which the differentiation is to be performed.

Internal forces, that is, forces exerted by parts of S on each other, are presumed to be such that their contributions to the generalized active forces F_{θ_k} and F_{q_r} can be expressed as

$$(F_{\theta_k})_I = 0 \qquad (k=1,2,3) \tag{5}$$

$$(F_{q_r})_I = -V(q)_{,r} - D_r(q,\dot{q}) \qquad (r=1,\ldots,n) \tag{6}$$

where $V(q)$, which has the character of a potential function, denotes a function of $q_1,\ldots,q_n$, whereas $D_r(q,\dot{q})$, which is associated with energy dissipation is a function of $q_1,\ldots q_n,\dot{q}_1,\ldots\dot{q}_n$, such that

$$D_r(q,0) = 0 \qquad (r=1,\ldots,n) \tag{7}$$

$$D_r(q,\dot{q})\ \dot{q}_r \geq 0 \tag{8}$$

Here, as in the remainder of the paper, the summation convention is used; that is, a repeated subscript index in a product indicates a summation in which this index takes on all values in its natural range (1,2,3 for k and ℓ, and $1,\ldots,n$ for r and s). If the equality sign in (10) holds only when $\dot{q}_1,\ldots,\dot{q}_n$ *all* vanish, the system S is said to have complete internal damping.

III. Stability Conditions

Stability conditions will be stated‡ for motions during which $\theta_k \equiv q_r \equiv 0$ ($k=1,2,3$; $r=1,\ldots,n$). Such motions are referred to as Earth-pointing because they occur when all particles and rigid bodies of S remain fixed in the orbital reference frame A. (Requiring the internal coordinates to vanish represents no restriction on the configuration of S, since coordinates that vanish when S assumes a particular configuration always can be found. As for the external coordinates, their vanishing implies that the central principal axes of S corresponding to $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ are, respectively, parallel to the local vertical, to the tangent to the orbit, and to the orbit normal.)

Speaking loosely, one can call an Earth-pointing motion stable if the system reacquires this motion, or at least acquires a motion closely resembling it, subsequent to every sufficiently small disturbance. To deal with this idea incisively, we define stability of an Earth-pointing motion in terms of the behavior of a point π with coordinates $\theta_k(t)$, $\dot{\theta}_k(t)$, $q_r(t)$, $\dot{q}_r(t)$ ($k=1,2,3$; $r=1,\ldots,n$) in a $2(n+3)$-dimensional vector space. An Earth-pointing motion is said to be stable if and only if there exists a neighborhood Q of the origin 0 of the space such that, if π lies in Q at $t=0$, then π remains in Q for all $t>0$. If, in addition, π approaches 0 as t approaches infinity, then the Earth-pointing motion is said to be asymptotically stable.

The formulation of the stability criteria to be presented requires the selection of a (not necessarily unique) Cartesian set of orthogonal coordinate axes, Z_1, Z_2, Z_3, originating at S^*, fixed in a reference frame Z, and oriented relative to S in such a way that two requirements are met: 1) the orientation of each axis depends uniquely on the values of $q_1,\ldots,q_n$, and 2) Z_1, Z_2, Z_3 are, respectively, parallel to $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ when $q_r=0$ ($r=1,\ldots,n$). Once Z_1, Z_2, and Z_3 have been selected, let $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, $\boldsymbol{z}_3$ be a dextral set of unit vectors, respectively, parallel to Z_1, Z_2, Z_3, and define $I_{jk}(q)$ as

$$I_{jk}(q) \triangleq \boldsymbol{z}_j \cdot \boldsymbol{J}(q) \cdot \boldsymbol{z}_k \qquad (j,k=1,2,3) \tag{9}$$

The formulation of stability criteria for an Earth-pointing motion now can be summarized as follows:

1) Select $q_1,\ldots,q_n$; form $V(q)$ [see Eq. (6)], and, using a tilde over a quantity to denote evaluation at $q_r=0$ ($r=1,\ldots,n$), compute $\tilde{V}_{,r}$ and $\tilde{V}_{,rs}$ ($r,s=1,\ldots,n$).

2) Identify an Earth-pointing motion of interest; designate the central principal axis of S which is parallel to the local vertical as 1, the principal axis parallel to the orbit tangent as 2, and the principal axis parallel to the orbit normal as 3; select Z_1, Z_2, Z_3; introduce $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, $\boldsymbol{z}_3$; and form $\tilde{I}_{11}$, $\tilde{I}_{22}$, $\tilde{I}_{33}$, $\tilde{I}_{jk,r}$, $\tilde{I}_{11,rs}$, $\tilde{I}_{22,rs}$, and $\tilde{I}_{33,rs}$ ($j,k=1,2,3$; $r,s=1,\ldots,n$) [see Eq. (9)].

3) Verify that the Earth-pointing motion under consideration can, in fact, occur by ascertaining that $q_r=0$ ($r=1,\ldots,n$) is a solution of the *internal equilibrium equations*

$$\tilde{V}_{,r} + \Omega^2\left(\tilde{I}_{11,r} - 2^{-1}\tilde{I}_{22,r} - \tilde{I}_{33,r}\right) = 0 \quad (r=1,\ldots,n) \tag{10}$$

4) Formulate the *external stability conditions:*

$$\tilde{I}_{33} - \tilde{I}_{22} > 0 \tag{11}$$

$$\tilde{I}_{22} - \tilde{I}_{11} > 0 \tag{12}$$

5) Form a symmetric $n \times n$ matrix $[\tilde{Z}_{,rs}]$ whose typical element $\tilde{Z}_{,rs}$ is defined as [see Ref. 1, Eq. (75)]

$$\tilde{Z}_{,rs} \triangleq \tilde{V}_{,rs} + \Omega^2\left[\tilde{I}_{11,rs} - 2^{-1}\tilde{I}_{22,rs} - \tilde{I}_{33,rs} - 3\,\frac{\tilde{I}_{12,r}\tilde{I}_{12,s}}{\tilde{I}_{22}-\tilde{I}_{11}} - 4\,\frac{\tilde{I}_{13,r}\tilde{I}_{13,s}}{\tilde{I}_{33}-\tilde{I}_{11}} - \frac{\tilde{I}_{23,r}\tilde{I}_{23,s}}{\tilde{I}_{33}-\tilde{I}_{22}}\right] \tag{13}$$

Apply Sylvester's criteria[22] to $[\tilde{Z}_{,rs}]$ to formulate n *internal stability conditions* [these conditions are analogous to Eq. (75) of Ref. 1]

$$\tilde{Z}_{,11} > 0, \quad \begin{vmatrix} \tilde{Z}_{,11} & \tilde{Z}_{,12} \\ \tilde{Z}_{,21} & \tilde{Z}_{,22} \end{vmatrix} > 0, \cdots, \quad \begin{vmatrix} \tilde{Z}_{,11} & \cdots & \tilde{Z}_{,1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \tilde{Z}_{,n1} & \cdots & \tilde{Z}_{,nn} \end{vmatrix} > 0 \tag{14}$$

6) Formulate the *pervasive damping conditions* (see Sec. 5 for a definition of pervasive damping). For many systems, the following procedure is useful in establishing explicit conditions sufficient to guarantee pervasive damping: Form $\tilde{K}_1$, $\tilde{K}_2$, $\tilde{K}_3$, and $C(\tilde{K}_1, \tilde{K}_2, \tilde{K}_3)$ as

$$\tilde{K}_1 \triangleq (\tilde{I}_{22} - \tilde{I}_{33})/\tilde{I}_{11} \tag{15}$$

$$\tilde{K}_2 \triangleq (\tilde{I}_{33} - \tilde{I}_{11})/\tilde{I}_{22} \tag{16}$$

$$\tilde{K}_3 \triangleq (\tilde{I}_{11} - \tilde{I}_{22})/\tilde{I}_{33} = -(\tilde{K}_1 + \tilde{K}_2)/(1 + \tilde{K}_1\tilde{K}_2) \tag{17}$$

$$C(\tilde{K}_1,\tilde{K}_2,\tilde{K}_3) \triangleq 9\tilde{K}_3^2 + 3(1 + 3\tilde{K}_2 - \tilde{K}_1\tilde{K}_2)\,\tilde{K}_3 - 4\tilde{K}_1\tilde{K}_2 \tag{18}$$

Determine in which region of the *external stability chart* (Fig. 1) the point $(\tilde{K}_1, \tilde{K}_2)$ lies. There are three such regions: $\tilde{K}_1 < 0$, $\tilde{K}_2 < 1$, $\tilde{K}_1 + \tilde{K}_2 < 0$; $\tilde{K}_1 < 1$, $\tilde{K}_2 < 0$, $\tilde{K}_1 + \tilde{K}_2 > 0$,

$(1+3\tilde K_2 - \tilde K_1 \tilde K_2)^2 - 16\tilde K_1 \tilde K_2 > 0$; and that portion of $-1 < \tilde K_i < 1\ (i=1,2)$ outside of the first two regions. Region 1 corresponds to that part of the $\tilde K_1$, $\tilde K_2$ space in which the external stability conditions are satisfied, whereas regions 2 and 3 contain points at which at least one of the external stability conditions is violated. The curve C_1 in Fig. 1 is the locus of points at which

$$C(\tilde K_1, \tilde K_2, \tilde K_3) = 0 \tag{19}$$

If none of the left-hand members in the inequalities (11, 12 or 14) vanish, the following three steps provide conditions sufficient for pervasive damping in regions 1 and 2 exclusive of the curve C_1:

6a) Form $D_1(q,\dot q),\ldots,D_n(q,\dot q)$, and verify that Eqs. (7) are satisfied. Form an $n\times n$ symmetric matrix $[\tilde D_{r,\dot s}]$ whose typical element $\tilde D_{r,\dot s}$ is the partial derivative of $D_r(q,\dot q)$ with respect to $\dot q_s$, evaluated at $q_r = \dot q_r = 0$ $(r=1,\ldots,n)$. Apply Sylevster's criteria to $[\tilde D_{r,\dot s}]$ to formulate *n internal damping conditions*

$$\tilde D_{1,\dot 1} > 0, \quad \begin{vmatrix} \tilde D_{1,\dot 1} & \tilde D_{1,\dot 2} \\ \tilde D_{2,\dot 1} & \tilde D_{2,\dot 2} \end{vmatrix} > 0, \cdots, \quad \begin{vmatrix} \tilde D_{1,\dot 1} & \cdots & \tilde D_{1,\dot n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \tilde D_{n,\dot 1} & & \tilde D_{n,\dot n} \end{vmatrix} > 0 \tag{20}$$

6b) Form the vector $Z_{\tilde h_r}$ as

$${}^Z\tilde h_r \triangleq \langle m\tilde p \times {}^Z\tilde p_{,r}\rangle \quad (r=1,\ldots,n) \tag{21}$$

where m and p are, respectively, the mass and the position vector from S^* to a generic particle of S. The angular brackets denote summation over all particles of S [see Eq. (27) if the system contains rigid bodies]. Form the quantities $\tilde H_{1r}$ and $\tilde H_{2r}$ given by

$$\tilde H_{1r} = Z_1\cdot Z_{\tilde h_r} - \tilde K_1^{-1}\,\tilde I_{23,r} \quad (r=1,\ldots,n) \tag{22}$$

$$\tilde H_{2r} = Z_2\cdot Z_{\tilde h_r} - \tilde K_2^{-1}\,\tilde I_{13,e} \quad (r=1,\ldots,n) \tag{23}$$

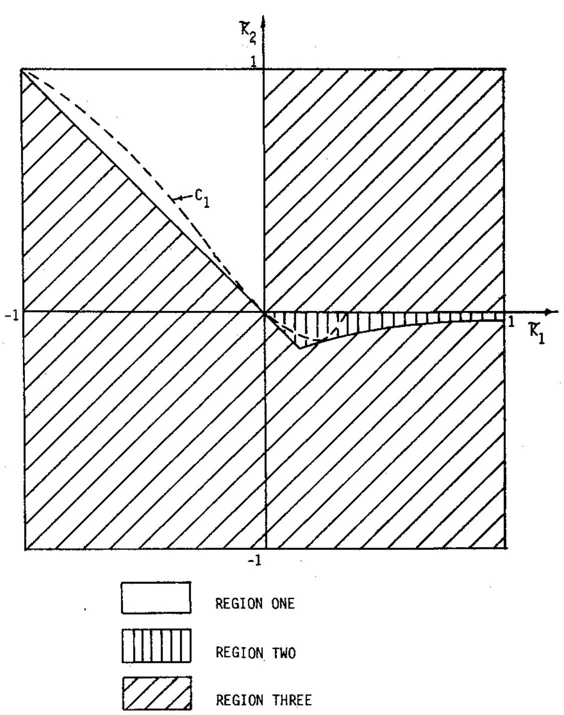

Fig. 1 External stability chart.

Any one of the following $2n$ conditions then is sufficient to insure *roll-yaw damping:*

$$\tilde H_{1r} \neq 0, \quad \tilde H_{2r} \neq 0 \quad (r=1,\ldots,n) \tag{24}$$

6c) Form $\tilde H_{3r}$ as

$$\tilde H_{3r} = Z_3\cdot Z_{\tilde h_r} - \tilde K_3^{-1}\,\tilde I_{12,r} \quad (r=1,\ldots,n) \tag{25}$$

Any one of the following $2n$ conditions is sufficient to insure pitch damping:

$$\tilde H_{3r} \neq 0, \ \tilde I_{33,r} \neq 0 \ (r=1,\ldots,n) \tag{26}$$

If all of the conditions from 6a, at least one of the conditions from 6b, and at least one of the conditions from 6c are satisfied, then the system is pervasively damped for regions 1 and 2 exclusive of the curve C_1.

When S contains one or more rigid bodies, a number of simplifications can be made in the foregoing procedure. For instance, the contribution of a typical such body, say R, to the quantities formed in step 2 frequently can be found most conveniently by the methods described in Sec. VIII of Ref. 1 [see Eqs. (76-81) of Ref. 1]. Furthermore, the contribution $(Z_{\tilde h_r})_R$ to $Z_{\tilde h_r}$ [see Eq. (23)] is given by

$$(Z_{\tilde h_r})_R = m_R \tilde r \times Z_{\tilde r,r} + \tilde R \cdot Z_{\tilde\omega,r}{}^R \quad (r=1,\ldots,n) \tag{27}$$

where m_R is the mass of R, r is the position vector relative to S^* of the mass center R^* of R, R is the inertia dyadic of R for R^*, and $Z_\omega{}^R$ is the angular velocity of R relative to Z.

When steps 1-6 have been taken, the following conclusions regarding the stability of an Earth-pointing motion can be drawn: the Earth-pointing motion is 1) stable if all conditions from 4 and 5 are satisfied; 2) asymptotically stable if all conditions from 4 and 5 are satisfied and the system is pervasively damped; and 3) unstable if any of the conditions from 4 or 5 are violated by reversal (that is, the left-hand member of the inequality is negative), and the system is pervasively damped.

It is worth noting that Fig. 1 is similar to the stability chart for the Earth-pointing motion of a rigid body in a circular orbit.[23] For the rigid body, one can show by means of a nonlinear analysis[24] that region 1 is associated with Liapunov stability; region 2 corresponds merely to infinitesimal stability (that is, stability as predicted by a study of linearized variational equations of motion); and region 3 corresponds to instability. For the flexible, dissipative satellite presently under consideration, region 1 is associated either with stable or asymptotically stable motion (assuming that the internal stability conditions are satisfied), and regions 2 and 3 correspond to instability if the system is pervasively damped. Thus, the character of region 2 changes with the addition of pervasive damping, an effect noted in various previous analyses (see, e.g., Refs. 2 and 25).

IV. Example

Steps 1-6 now will be carried out for the skewed boom satellite shown in Fig. 2. This satellite is of practical interest, and, having but one internal degree of freedom, it can be analyzed with a minimum of mathematical complexity and is thus well suited as an illustrative example:

1) In Fig. 2, S^* designates the common mass center of two rigid bodies, X and Y, which together form a system S; x_1, x_2,

and $x_3 = x_1 \times x_2$ are unit vectors parallel to the central principal axes of X; y_1, y_2, and $y_3 = y_1 \times y_2$ are unit vectors parallel to the central principal axes of Y; and q_1, which plays the part of (the only) internal coordinate, measures the angle between x_1 and y_1. The bodies X and Y can rotate relative to each other about an axis passing through S^*, normal to x_1 and y_1, and forming an acute angle α with lines parallel to x_2 and y_2, so that $x_i = y_i$ $(i=1,2,3)$ when $q_1 = 0$. Such rotations are resisted by a torsion spring and a damper, which exert torques of magnitude $k_1 |q_1|$ and $\mu_1 |\dot{q}_1|$, respectively, on each of the bodies, where k_1 and μ_1 are certain constants. The contribution $(F_{q_1})_1$ of these torques to the generalized active force F_{q_1} is given by $(F_{q_1})_1 = -k_1 q_1 - \mu_1 \dot{q}_1$. Hence, if $V(q)$ and $D_1(q,\dot{q})$ are defined as

$$V(q) \triangleq \tfrac{1}{2} k_1 q_1^2 \tag{28}$$

$$D_1(q,\dot{q}) \triangleq \mu_1 \dot{q}_1 \tag{29}$$

then $(F_{q_1})_1$ can be expressed as in Eq. (6), with $D_1(q,\dot{q})\dot{q}_1 = \mu_1 \dot{q}_1^2$, which shows that Eqs. (7) are satisfied and that the inequality (8) is satisfied if $\mu_1 \geq 0$.

2) The Earth-pointing motion of interest is described by taking $q_1 = 0$ and requiring both that S^* move in a circular orbit in accordance with the description in Sec. 2 and that x_1, x_2, x_3 (and hence y_1, y_2, y_3) remain parallel to $E - S^*$, to the tangent to the orbit, and to the orbit normal, respectively. For Z_1, Z_2, Z_3, one may take the principal axes of X for X^*. Letting X_1, X_2, X_3 and Y_1, Y_2, Y_3 denote, respectively, the central principal moments of inertia of X and Y associated with x_1, x_2, x_3 and y_1, y_2, y_3, one then finds that

$$\tilde{I}_{11} = X_1 + Y_1, \quad \tilde{I}_{22} = X_2 + Y_2, \quad \tilde{I}_{33} = X_3 + Y_3 \tag{30}$$

$$\tilde{I}_{11,1} = \tilde{I}_{22,1} = \tilde{I}_{33,1} = 0, \quad \tilde{I}_{12,1} = (Y_1 - Y_2) s_\alpha \tag{31}$$

$$\tilde{I}_{13,1} = (Y_3 - Y_1) c_\alpha, \quad \tilde{I}_{23,1} = 0 \tag{32}$$

$$\tilde{I}_{11,11} = 2(Y_2 s_\alpha^2 + Y_3 c_\alpha^2 - Y_1) \tag{33}$$

$$\tilde{I}_{22,11} = 2(Y_1 - Y_2) s_\alpha^2, \quad \tilde{I}_{33,11} = 2(Y_1 - Y_3) c_\alpha^2 \tag{34}$$

where s_α and c_α denote $\sin\alpha$ and $\cos\alpha$, respectively.

3) Since $\tilde{V}_{,1} = \tilde{I}_{11,1} = \tilde{I}_{22,1} = \tilde{I}_{33,1} = 0$, Eq. (10) is satisfied, and the postulated motion can occur.

4) In view of Eqs. (6), the external stability conditions are

$$\beta_X + \beta_Y - \gamma_X - \gamma_Y > 0 \tag{35}$$

$$\gamma_X + \gamma_Y > 0 \tag{36}$$

where β_X, β_Y, γ_X, and γ_Y are defined as

$$\beta_X \triangleq X_3 - X_1, \quad \beta_Y \triangleq Y_3 - Y_1,$$
$$\gamma_X \triangleq X_2 - X_1, \quad \gamma_Y \triangleq Y_2 - Y_1.$$

5) Substituting from Eqs. (28 and 30-34) into Eq. (13), one obtains

$$\tilde{Z}_{,11} = k_1 + \Omega^2 \{ 3 s_\alpha^2 [\gamma_X \gamma_Y / (\gamma_X + \gamma_Y)] + 4 c_\alpha^2 [\beta_X \beta_Y / (\beta_X + \beta_Y)] \}$$

To satisfy the inequalities (14), it is thus necessary and sufficient that

$$k_1 \Omega^{-2} + 3 s_\alpha^2 [\gamma_X \gamma_Y / (\gamma_X + \gamma_Y)] + 4 c_\alpha^2 [\beta_X \beta_Y / (\beta_X + \beta_Y)] > 0 \tag{37}$$

In accordance with step 1 of Sec. 3, the Earth-pointing motion under consideration is stable if the inequalities (35-37) are all satisfied.

6) Substitution from Eqs. (30) into Eqs. (15-17) yields

$$\tilde{K}_1 = \frac{X_2 - X_3 + Y_2 - Y_3}{X_1 + Y_1}, \quad \tilde{K}_2 = \frac{X_3 - X_1 + Y_3 - Y_1}{X_2 + Y_2} \tag{38}$$

$$\tilde{K}_3 = \frac{X_1 - X_2 + Y_1 - Y_2}{X_3 + Y_3} \tag{39}$$

and these quantities may be substituted into Eq. (18) to determine whether or not one is dealing with a point on the curve C_1 in Fig. 1.

6a) From Eq. (29) and the inequalities (20), it follows that the internal damping condition is satisfied if

$$\mu_1 > 0 \tag{40}$$

6b) Since X and Y are rigid bodies, their contributions to $Z_{\tilde{h}_1}$ may be calculated by using Eq. (27). The angular velocities $Z_\omega{}^X$ and $Z_\omega{}^Y$ are given by $Z_\omega{}^X = 0$ $z_\omega{}^Y = \dot{q}_1 (c_\alpha z_2 + s_\alpha z_3)$. Hence, with $r^X = r^Y = 0$, one finds that

$$z_{\tilde{h}_1} = Y_2 c_\alpha z_2 + Y_3 s_\alpha z_3 \tag{41}$$

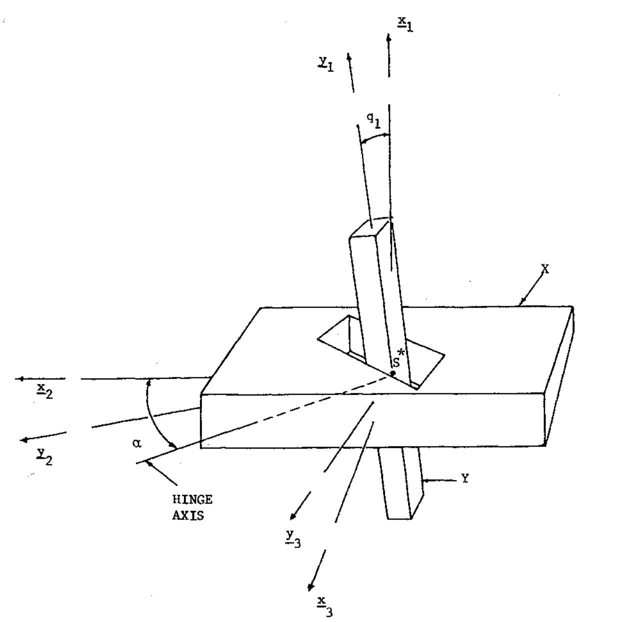

Fig. 2 Skewed boom satellite.

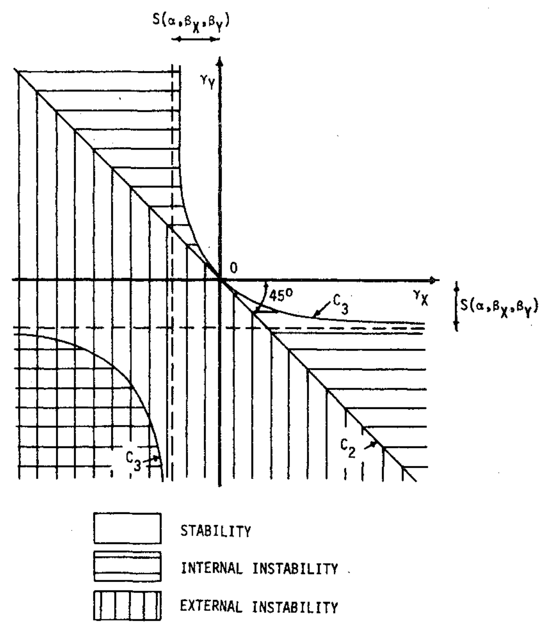

Fig. 3 Stability chart for skewed boom satellite $[S(\alpha, \beta_X, \beta_Y) > 0$ $\beta_X + \beta_Y - \gamma_X - \gamma_Y > 0]$.

and, using Eqs. (31, 32, 38, 39, 41, 22, and 23), one obtains $\tilde{H}_{11}=0$ and

$$\tilde{H}_{21}=\left[\frac{Y_2(X_3-X_1)-X_2(Y_3-Y_1)}{X_3-X_1+Y_3-Y_1}\right]c_\alpha$$

The roll-yaw damping conditions (24) thus are satisfied if $c_\alpha \neq 0$ and

$$(X_3-X_1)/X_2 \neq (Y_3-Y_1)/Y_2 \tag{42}$$

6c) Proceeding similarly, one finds that the pitch damping conditions (26) are satisfied if $s_\alpha \neq 0$ and

$$(X_2-X_1)/X_3 \neq (Y_2-Y_1)/Y_3 \tag{43}$$

If (40, 42, and 43) are satisfied, and if α is not equal to 0 or $\pi/2$ (that is, if the hinge axis is skewed relative to the principal axes of the satellite), then the system is pervasively damped for regions 1 and 2 of the external stability chart exclusive of curve C_1. If, in addition, the inequalities (35-37) are satisfied, then, by step 2 of Sec. 3, the Earth-pointing motion under consideration is asymptotically stable; but if at least one of these inequalities is violated by reversal and the motion is pervasively damped, then, in accordance with step 3 of Sec. 3, the motion is unstable. These results are illustrated in Fig. 3, where C_2 and C_3 are curves whose equations are, respectively, $\gamma_X+\gamma_Y=0$ and $(\gamma_X+\gamma_Y)S(\alpha,\beta_X,\beta_Y)+\gamma_X\gamma_Y=0$, with

$$S(\alpha,\beta_X,\beta_Y) \triangleq [k\Omega^{-2}+4c_\alpha^2\beta_X\beta_Y(\beta_X+\beta_Y)^{-1}](3s_\alpha^2)^{-1}$$

Figure 3 corresponds to systems for which $S(\alpha,\beta_X,\beta_Y)>0$, there is a similar stability chart (not included) corresponding to systems for which $S(\alpha,\beta_X,\beta_Y)<0$. The curve C_2 separates a region in the γ_X,γ_Y plane associated with external instability from one corresponding to external stability. Both C_2 and C_3 separate regions in the γ_X,γ_Y plane associated with internal instability from ones corresponding to internal stability. Thus, the points in this plane lying in the unshaded portion represent values of the system parameters for which the motion is stable, provided that the inequality (35) is satisfied. If the system is pervasively damped, then the unshaded portion corresponds to asymptotically stable configurations [provided that the inequality (35) is satisfied], and the shaded portion corresponds to ustable configurations.

One interesting feature of the stability chart is that there are regions where the external stability conditions are satisfied but the internal stability conditions are not satisfied. This can occur for $k_1>0$, demonstrating that the stability conditions for the flexible, Earth-pointing satellite are not merely a simple combination of the stability conditions for a rigid body (with the same mass distribution as the flexible satellite in its nominal configuration) and the Dirichlet conditions for the internal potential energy of the satellite.

V. Theoretical Basis

Letting T denote the kinetic energy of S in N, one can express Lagrange's equations of motion of S as

$$\dot{T}_{,\dot{k}}-T_{,k}=F_{\theta_k} \quad (k=1,2,3) \tag{44}$$

$$\dot{T}_{,\dot{r}}-T_{,r}=F_{q_r} \quad (r=1,\ldots,n) \tag{45}$$

where $T_{,\dot{r}}$ and $T_{,\dot{k}}$ denote, respectively, the partial derivatives of T with respect to $\dot{\theta}_k$ and $\dot{q}_r$ $(k=1,2,3;\ r=1,\ldots,n)$. The generalized active forces F_{θ_k} and F_{q_r} are given by

$$F_{\theta_k}=(F_{\theta_k})_E+(F_{\theta_k})_I \quad (k=1,2,3)$$

$$F_{q_r}=(F_{q_r})_E+(F_{q_r})_I \quad (r=1,\ldots,n)$$

and detailed expressions for the terms appearing in the right-hand members of these equations are available in Eqs. (1, 2, 5, and 6).

In the following, we shall be concerned solely with linearized equations of motion. To generate these by substitution into Eqs. (44) and (45), one can use for T an expression in which terms of third and higher degree in θ_k, $\dot{\theta}_k$, q_r, and $\dot{q}_r$ $(k=1,2,3;\ r=1,\ldots,n)$ have been dropped. Moreover, if one defines $g_{rs}(q)$, $B_{h_r}(q)$, and $H_{kr}(q)$ as

$$g_{rs}(q) \triangleq \langle mB_{p,r}\cdot B_{p,s}\rangle \quad (r,s=1,\ldots,n) \tag{46}$$

$$B_{h_r}(q) \triangleq \langle mp\times B_{p,r}\rangle \quad (r=1,\ldots,n) \tag{47}$$

$$H_{kr}(q) \triangleq b_k\cdot B_{h_r}(q) \quad (k=1,2,3;\ r=1,\ldots,n) \tag{48}$$

where the angular brackets denote summation over all particles of S, then, by methods similar to those employed in Ref. 1, one can verify that the quadratic approximation to T is given by

$$T=T_i+T_1+T_2 \tag{49}$$

where, with $\tilde{T}_0$ denoting the part of T that is independent of θ, $\dot{\theta}$, q and $\dot{q}$

$$T_0 \triangleq \tilde{T}_0+2^{-1}\Omega^2[\tilde{J}_{3,r}q_r+(\tilde{J}_2-\tilde{J}_3)\theta(\tilde{J}_2-\tilde{J}_3)\theta_1^2+(\tilde{J}_1-\tilde{J}_3)\theta_3^2$$

$$T_1 \triangleq \Omega[\tilde{J}_3\dot{\theta}_3+\tilde{H}_{3r}\dot{q}_r+(\tilde{J}_3-\tilde{J}_1)\theta_2\dot{\theta}_1+\tilde{J}_2\theta_1\dot{\theta}_2$$
$$+\tilde{J}_{3,r}q_r\dot{\theta}_3+(\tilde{H}_{2r}\theta_1-\tilde{H}_{1r}\theta_2)\dot{q}_r+\tilde{H}_{3r,s}q_s\dot{q}_r]$$

$$T_2 \triangleq (\tilde{J}_1\dot{\theta}_1^2+\tilde{J}_2\dot{\theta}_2^2+\tilde{J}_3\dot{\theta}_3^2)+2^{-1}\tilde{g}_{rs}\dot{q}_r\dot{q}_s$$
$$+(\tilde{H}_{1r}\dot{\theta}_1+\tilde{H}_{2r}\dot{\theta}_2+\tilde{H}_{3r}\dot{\theta}_3)\dot{q}_r$$

Equations of motion in which only linear terms appear explicity now can be generated by using T as given in Eq. (49) when carrying out the differentiations indicated in the left-hand members of (1) and (2), while using linearized expressions for F_{θ_r} and F_{q_r} for the right-hand members. When this is done, one finds that the equations of motion are satisfied by $\theta=q=0$ if Eq. (7) is satisfied and if

$$\tilde{Z}_{,r}=0 \quad (r=1,\ldots,n) \tag{50}$$

where

$$Z(q) \triangleq V(q)+\Omega^2[J_1(q)-2^{-1}J_2(q)-J_3(q)]$$

It can be shown that Eq. (50) is equivalent to Eq. (10).

When Eqs. (7) and (50) are satisfied, the equations of motion of S can be written as the matrix equation

$$M\ddot{x}+G\dot{x}+Kx=\begin{bmatrix}0\\0\\0\\-D_1(q,\dot{q})\\ \cdot\\ \cdot\\ -D_n(q,\dot{q})\end{bmatrix}+\begin{bmatrix}0_2(\theta,\dot{\theta},q,\dot{q})\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ 0_2(\theta,\dot{\theta},q,\dot{q})\end{bmatrix} \tag{51}$$

where x, M, G, and K are defined as follows:

$$x \triangleq [\theta_1\theta_2\theta_3q_1\ldots q_n]^T$$

$$M \triangleq \begin{bmatrix} \tilde{J}_1 & 0 & 0 & \tilde{H}_{11} & \cdots & \tilde{H}_{1n} \\ 0 & \tilde{J}_2 & 0 & \tilde{H}_{21} & \cdots & \tilde{H}_{2n} \\ 0 & 0 & \tilde{J}_3 & \tilde{H}_{31} & \cdots & \tilde{H}_{3n} \\ \tilde{H}_{11} & \tilde{H}_{21} & \tilde{H}_{31} & \tilde{g}_{11} & \cdots & \tilde{g}_{1n} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \tilde{H}_{1n} & \tilde{H}_{2n} & \tilde{H}_{3n} & \tilde{g}_{n1} & \cdots & \tilde{g}_{nn} \end{bmatrix}$$

$$G \triangleq \Omega \begin{bmatrix} 0 & (\tilde{J}_3-\tilde{J}_1-\tilde{J}_2) & 0 & -\tilde{H}_{21} & \cdots & -\tilde{H}_{2n} \\ -(\tilde{J}_3-\tilde{J}_1-\tilde{J}_2) & 0 & 0 & \tilde{H}_{11} & \cdots & \tilde{H}_{1n} \\ 0 & 0 & 0 & \tilde{J}_{3,1} & \cdots & \tilde{J}_{3,n} \\ \tilde{H}_{21} & -\tilde{H}_{11} & -\tilde{J}_{3,1} & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & (\tilde{H}_{3r,s}-\tilde{H}_{3s,r}) & \\ \cdot & \cdot & \cdot & & & \\ \tilde{H}_{2n} & -\tilde{H}_{1n} & -\tilde{J}_{3,n} & & & \end{bmatrix}$$

$$K \triangleq \Omega^2 \begin{bmatrix} (\tilde{J}_3-\tilde{J}_2) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4(\tilde{J}_3-\tilde{J}_1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3(\tilde{J}_2-\tilde{J}_1) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \tilde{Z}_{,11} & \cdots & \tilde{Z}_{,1n} \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \tilde{Z}_{,n1} & \cdots & \tilde{Z}_{,nn} \end{bmatrix}$$

The matrix M is a symmetric, positive definite "inertia" matrix; G is a skew-symmetric "gyroscopic" matrix; and K is a symmetric "stiffness" matrix.

To construct a Liapunov function, we begin by premultiplying Eq. (51) with $\dot{x}^T$, which yields

$$2\dot{H} = -D_r \dot{q}_r \tag{52}$$

where

$$H \triangleq 2^{-1}(\dot{x}^T M\dot{x} + x^T K x) + 0_3(\theta,\dot{\theta},q,\dot{q})$$
$$= 2^{-1}[(\dot{x}^T M \dot{x} + (\tilde{J}_3-\tilde{J}_2)\theta_1^2 + 4(\tilde{J}_3-\tilde{J}_1)\theta_2^2 + 3(\tilde{J}_2-\tilde{J}_1)\theta_3^2 + \tilde{Z}_{,rs}q_r q_s] + 0_3(\theta,\dot{\theta},q,\dot{q})$$

(The quantity H is the Hamiltonian of the conservative system formed by setting $D_1 = \ldots = D_n = 0$.) Stability criteria can be established by stating conditions under which H is a Liapunov function. To this end, note first that, as a consequence of Eqs. (52) and (8), $\dot{H} \leq 0$. Next, suppose that

$$\tilde{J}_3 - \tilde{J}_2 > 0 \tag{53}$$

$$\tilde{J}_2 - \tilde{J}_1 > 0 \tag{54}$$

and that there exists a neighborhood of 0 [in the $2(n+3)$ dimensional vector space described in Sec. 3] at every point of which

$$\tilde{Z}_{,rs} q_r q_s > 0 \tag{55}$$

when q does not vanish identically. Then there exist neighborhoods of 0 in which the sign of H depends solely on the quadratic portion of H; H is a positive definite function of its arguments in every such neighborhood (recall that M is positive definite so that $\dot{x}^T M \dot{x}$ is nonnegative); and H is thus a Liapunov function. Accordingly,[26] when the three conditions (53-55) are satisfied, the Earth-pointing motion described by $\theta=\dot{\theta}=q=\dot{q}=0$ is stable. Moreover, by adding a fourth requirement, one can guarantee asymptotic stability. This requirement is that there exist a neighborhood of 0 in which 0 is the only point for which both $\dot{H}=0$ and Eq. (51) are satisfied. We then say that for the Earth-pointing motion under consideration the system is pervasively damped, and we may conclude that the motion is asymptotically stable.

By Sylvester's criteria, condition (55) is satisfied if and only if all principal minors of $[\tilde{Z}_{,rs}]$ are positive, that is, if and only if the internal stability conditions presented in Sec. 3 are satisfied. Thus, if at least one of the inequalities (53, 54, or 14) is violated by reversal, then there exists in every neighborhood of 0 points at which H is negative. If, in addition, the damping is pervasive, then the motion under consideration is unstable (Ref. 26, p. 38).

To obtain conditions that guarantee pervasive damping,[§] we begin by expanding $\tilde{D}_r(q,\dot{q})$ as

$$D_r(q,\dot{q}) = \tilde{D}_r + \tilde{D}_{r,s}q_s + \tilde{D}_{r,\dot{s}}\dot{q}_s + 0_2(q,\dot{q}) \tag{56}$$

and note that, if $[\tilde{D}_{r,\dot{s}}]$ is positive definite, then it follows from Eq. (52) that $\dot{H}\equiv 0$ implies $\dot{q}_r \equiv 0$ $(r=1,\ldots,n)$. (The identity symbol here denotes equality for all t.) Under these circumstances, one immediately has $\ddot{q}_r \equiv 0$ $(r=1,\ldots,n)$, and the linear portion of Eq. (51) yields

$$\tilde{J}_1\ddot{\theta}_1 + \Omega(\tilde{J}_3-\tilde{J}_1-\tilde{J}_2)\dot{\theta}_2 + \Omega^2(\tilde{J}_3-\tilde{J}_2)\theta_1 = 0 \tag{57}$$

$$\tilde{J}_2\ddot{\theta}_2 - \Omega(\tilde{J}_3-\tilde{J}_1-\tilde{J}_2)\dot{\theta}_1 + 4\Omega^2(\tilde{J}_3-\tilde{J}_1)\theta_2 = 0 \tag{58}$$

$$\tilde{J}_3\ddot{\theta}_3 + \Omega^2(\tilde{J}_2-\tilde{J}_1)\theta_3 = 0 \tag{59}$$

§The approach here taken is due to Pringle.[30] An alternative approach may be found in Ref. 27.

$$\begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{21} & \tilde{H}_{31} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \tilde{H}_{1n} & \tilde{H}_{2n} & \tilde{H}_{3n} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \\ \ddot{\theta}_2 \\ \\ \ddot{\theta}_3 \end{bmatrix} + \Omega \begin{bmatrix} \tilde{H}_{21} & -\tilde{H}_{11} & -\tilde{J}_{3,1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \tilde{H}_{2n} & -\tilde{H}_{1n} & -\tilde{J}_{3,n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \\ \dot{\theta}_2 \\ \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \tilde{Z}_{,11} & \cdots & \tilde{Z}_{,1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \tilde{Z}_{,n1} & \cdots & \tilde{Z}_{,nn} \end{bmatrix} \begin{bmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{bmatrix} = 0 \qquad (60)$$

Thus, the three quantities θ_1, θ_2, θ_3 must satisfy $n+3$ differential equations. One way that they can do so is by vanishing identically, provided that $q_r=0$ $(r=1,...,n)$. To make certain that there is no other way, one can require that at least one of $\tilde{H}_{1r}$, $\tilde{H}_{2r}$ $(r=1,...,n)$ and one of H_{3r}, $J_{3,r}$ $(r=1,...,n)$ differ from zero. If none of the left-hand members of the internal stability conditions (14) vanish, then $[\tilde{Z}_{,rs}]$ is nonsingular, and it is then possible to solve Eqs. (60) for $q_1,...,q_n$. Such solution leads to time-dependant expressions for $q_r(r=1,...,n)$ unless $\theta_i \equiv 0$ $(i=1,2,3)$, since any solution of Eqs. (57-59), other than the null solution, generally involves trigonometric functions variously possessing three distinct periods. But $\dot{H}\equiv 0$ implies that $\dot{q}_r \equiv 0$ $(r=1,...,n)$, as already mentioned. Hence $q_r(r=1,...,n)$ cannot be time-dependent, and $\theta_i \equiv 0$ $(i=1,2,3)$, $q_r \equiv 0$ $(r=1,...,n)$ is now the only possibility. Consequently, pervasive damping is guaranteed, except when the period of θ_3 [see Eq. (59)] is equal to one of the two periods associated with the solution of Eqs. (57) and (58). This is the reason for taking the curve C_1 defined in Eq. (19) into account[22] explicitly when speaking of pervasive damping.

It remains to be shown that $\tilde{H}_{kr}(k=1,2,3; r=1,...,n)$ can be expressed as in Eqs. (22, 23, and 25). To this end, note first that one may write

$$B_{p,r} = Z_{\bar{p},r} - Z_{\bar{\omega},rr} \times \bar{p} \quad (r=1,...,n)$$

where ω is the angular velocity of B in Z. Substitution into Eq. (47) and use of Eq. (21) then gives

$$B_{\bar{h}_r} = Z^{\bar{h}_r} - \langle m\bar{p} \times (Z_{\bar{\omega},r} \times \bar{p}) \rangle \quad (r=1,...,n)$$

and, after using the identities

$$\langle m\bar{p} \times Z_{\bar{\omega}\dot{r}} \times \bar{p}) \rangle \equiv \langle m(\bar{p}^2 U - \bar{p}\bar{p}) \rangle \cdot Z_{\bar{\omega},\dot{r}} \equiv \bar{J} \cdot Z_{\bar{\omega},\dot{r}}$$

where U is the unit dyadic, one arrives at

$$B_{\bar{h}_r} = Z_{\bar{h}_r} - \bar{J} \cdot Z_{\bar{\omega},\dot{r}} \quad (r=1,2,...,n) \qquad (61)$$

Next, the angular velocity ω can be written as[28]

$$\omega = z_{\dot{b}_2} b_3 b_1 + z_{\dot{b}_3} \cdot b_1 b_2 + z_{\dot{b}_1} \cdot b_2 b_3 \qquad (62)$$

where Z_{b_k} is the time derivative in A of b_k $(k=1,2,3)$, and, if functions $b_{jk}(q)$ are defined as $b_{jk}(q) \triangleq b_j \circ z_k$ so that $\bar{b}_{jk} = \delta_{jk}$, then the right-hand side of the identity $b_j \equiv b_{jk} z_k$ can be expanded in a power series in q to obtain

$$b_j = z_j + \bar{b}_{jk,r} q_r z_k + 0_2(q) \qquad (j=1,2,3) \qquad (63)$$

This expression for b_j yields, upon differentiation with respect to time in reference frame Z,

$$z_{\dot{b}_j} = \bar{b}_{jk,r} \dot{q}_r z_k + 0_2(q,\dot{q}) \qquad (j=1,2,3) \qquad (64)$$

Substitution from Eqs. (63) and (64) into Eq. (62) and subsequent differentiation with respect to $\dot{q}_r$ then shows that

$$Z_{\bar{\omega},\dot{r}} = \bar{b}_{23,r} z + \bar{b}_{31,r} z_2 + \bar{b}_{12,r} z_3 \qquad (r=1,...,n) \qquad (65)$$

Finally, it may be verified[29] that

$$\bar{b}_{jk,r} = (\bar{I}_{jk,r} - \bar{I}_{jj,r}\delta_{jk}) / (\bar{I}_{jj} - \bar{I}_{kk}) \qquad (j,k=1,2,3;\ r=1,\ ...,\ n)$$

and use of these expressions in conjunction with Eqs. (65, 4, 15-17) permits one to rewrite Eq. (61) as

$$B_{\bar{h}_r} = Z_{\bar{h}_r} - \bar{K}_1^{-1} \bar{I}_{23,r} z_1 - \bar{K}_2^{-1} \bar{I}_{13} \bar{I}_{13,r} z_2 - \bar{K}_3^{-1} \bar{I}_{12,r} z_3$$

from which the validity Eqs. (22, 23, and 25) follows by substitution into Eq. (48).

VI Conclusions

The procedure set forth in Sec. 3 has been applied to a number of satellite configurations having up to six internal degrees of freedom. In each instance, there was a substantial savings in labor resulting from uncoupling external and internal stability conditions. It may appear that the savings in labor become less and less significant as the number of internal degrees of freedom increases. Actually, the opposite is the case, for the labor in question arises in connection with testing a matrix for positive definiteness. Hence, one should compare testing a matrix of order n with testing one of order $n+3$. The latter requires evaluation of only three additional determinants, but each of these is larger than the largest one considered when one works with a matrix of order n, and the additional labor required thus grows with n.

References

[1] Teixeira-Filho, D. R. and Kane, T. R., "Spin Stability of Torque-Free Systems—Part 1," *AIAA Journal*, Vol. 11, June 1973, pp. 862-867.

[2] Pringle, R., "On the Capture, Stability, and Passive Damping of Artificial Satellites," Ph. D. Dissertation, 1964, Stanford University, also SUDAER 181, 1964, Dept. of Aeronautics and Astronautics, Stanford University.

[3] Pringle, R., "On the Stability of a Body with Connected Moving Parts," *AIAA Journal*, Vol. 4, Aug. 1966, pp. 1395-1404.

[4] Pringle, R., Jr., "Stability of Force-Free Motion of a Dual Spin Spacecraft," *AIAA Journal*, Vol. 7, June 1969, pp. 1054-1063.

[5] Willems, P. Y., "Stability of Deformable Gyrostats on a Circular Orbit," *Journal of the Astronautical Sciences*, Vol. XVIII, Feb. 1970, pp. 65-85.

[6] Willems, P. Y., "Attitude Stability of Deformable Satellites," *Evolution D'Attitude et Stabilization des Satellites, CNES International Symposium*, Oct. 8-11, 1968, Paris.

[7] Buckins, F., Gorez, R., and Willems, P. Y., "Influence of Structural Elasticity of Attitude, Stability and Drift of Spacecraft," Final Scientific Rept., Grant AF EOAR 65-66, July 1967.

[8] Meirovitch, L., "Stability of a Spinning Body Containing Elastic Parts via Liapunov's Direct Method," *AIAA Journal*, Vol. 8, July 1970, pp. 1193-1200.

[9] Meirovitch, L., "Liapunov Stability Analysis of Hybrid Dynamical Systems with Multi-Elastic Domains," *International Journal of Nonlinear Mechanics*, Vol. 281, Jan. 1966, pp. 51-72.

[10] Wang, P. K. C., "Stability Analysis of Elastic and Aeroelastic Systems via Lyapunov's Direct Method," *Journal of the Franklin Institute*, Vol. 281, Jan. 1966, pp. 51-72.

[11] Bainum, P. M. and Mackison, D. L., "Gravity-Gradient Stabilization of Synchronous Orbiting Satellites," *Journal of the British Interplanetary Society*, Vol. 21, 1968, pp. 341-369.

[12] Tinling, B. E. and Merrick, V. K., "Exploitation of Inertial Coupling in Passive Gravity-Gradients-Stabilized Satellites," *Journal of Spacecraft and Rockets*, Vol. 1, July-Aug. 1964, pp. 381-387.

[13] Nelson, H. D. and Meirovitch, L., "Stability of a Non-Symmetrical Satellite with Elastically Connected Moving Parts," *Journal of the Astronautical Sciences*, Vol. XIII, Nov.-Dec. 1966, pp. 226-234.

[14] Bainum, P. M., Harkness, R. E., and Stuiver, W., "Attitude Stability and Damping of a Tethered Orbiting Interferometer Satellite System," *Journal of the Astronautical Sciences*, Vol. XIX, March-April 1972, pp. 364-389.

[15] Robe, T. R., "Stability of Two Tethered Unsymmetrical Earth-Pointing Bodies," *AIAA Journal*, Vol. 6, Dec. 1966, pp. 2282-2288.

[16] Routh, E. J., *Advanced Dynamics of a System of Rigid Bodies*, 6th ed., Dover, New York, 1955, pp. 223-231.

[17] Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Springer-Verlag, Berlin, 1959, pp. 21 and 34.

[18] Fletcher, H. J., Rongved, L., and Yu, E. V., "Dynamics Analysis of a Two Body Gravitationally Oriented Satellite," *The Bell System Technical Journal*, Vol. 42, Sept. 1963, pp. 2237-2266.

[19] Cronin, R. H. and Kane, T. R., "Gravitational Stabilization of a Two-Body Satellite," *Journal of Spacecraft and Rockets*, Vol. 10, May 1973, pp. 291-294.

[20] Porter, B., *Stability Criteria for Linear Dynamical Systems*, Academic Press, New York, 1968, pp. 74-82.

[21] Plummer, H. C., *An Introductory Treatise on Dynamical Astronomy*, Dover, New York, 1960, p. 294.

[22] Hahn, W., *Stability of Motion*, Springer-Verlag, New York, 1967, p. 100.

[23] DeBra, D. B. and Delp, R. H., "Rigid Body Attitude Stability and Natural Frequencies in a Circular Orbit," *Journal of the Astronautical Sciences*, Vol. VIII, Jan. 1961, pp. 14-17.

[24] Beletskii, V. V., "The Librations of a Satellite," *Artificial Earth Satellites*, Vol. 3, edited by L. V. Kurnosova, Plenum Press, New York, 1961, pp. 18-45.

[25] Kane, T. R. and Barba, P. M., "Effects of Energy Dissipation on a Spinning Satellite," *AIAA Journal*, Vol. 4, Aug. 1966, pp. 1391-1394.

[26] LaSalle, J. and Lefschetz, S., *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961, p. 37.

[27] Connell, G.M., "Asymptotic Stability of Second Order Linear Systems with Semidefinite Damping," *AIAA Journal*, Vol. 7, June 1969, pp. 1185-1187.

[28] Kane, T.R., *Dynamics*, Holt, Rinehart and Winston, New York, 1968, p. 21.

[29] Kane, T.R., "Principal Axes and Moments of Inertia of Deformable Systems," *Proceedings of the 8th Aerospace Mechanisms Symposium*, 1973.

[30] Pringle, R., private communication. Dec. 1974, Institute For Defense Analysis, Arlington Va.